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ACCELERATION WAVES IN ELASTO-PLASTICITY

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Abstract—A spectral analysis of the acceleration wave problem in general elasto-plastic materials is carried out, whereby explicit expressions for the eigenvalues and eigenvectors are obtained. In case of nonassociated plasticity, all eigenvectors become nonorthogonal and one eigenvalue always remains unchanged and equal to the shear modulus. For a very broad class of nonassociated plasticity models, it is shown that the eigenvalues are always real, implying that so-called "divergence" instability can occur, while "flutter" instability can never occur. It is found that a certain value of the hardening modulus exists for which specific propagation directions will always imply that all wave speeds are identical and equal to the elastic distortion wave speed. Moreover, in this situation the eigenvectors are arbitrary, corresponding to a state of diffuse wave modes. The criteria of von Mises and Rankine are used to illustrate some of the findings.

INTRODUCTION

Propagation of acceleration waves in solid bodies is a phenomenon which has long been subject to intensive investigation. It turns out that the fundamental nature of acceleration waves relates directly to the important issues of stability, static bifurcations and plane wave propagation. Moreover, recent finite element developments to capture the localization of strains in thin zones within a body have intensified the interest in acceleration waves considerably.

The pioneering work by Hadamard (1903), where elastic bodies were studied, established the basis for the analysis. Hill (1961, 1962) and Mandel (1962, 1964) extended this work to elasto-plasticity and further progress was obtained by Rice (1976). The work of Truesdell (1965) provides a comprehensive treatment of many of the aspects related to acceleration waves. Hill (1962) determined analytical expressions by which the eigenvalues and eigenvectors for associated plasticity can be determined. However, except for the advances made by Mandel (1964), the issue of how to determine the eigenvalues and eigenvectors for general nonassociated plasticity has been left open, which means that it has been difficult to gain insight into the fundamental mechanisms of different types of material models. The eigenvalues and eigenvectors can always be determined numerically for different wave propagation directions, cf. Sobh (1987), but apart from being time-consuming, such an approach does not facilitate a concise and general evaluation. Here, we shall present explicit analytical expressions for the eigenvalues and eigenvectors for general nonassociated plasticity. We shall thereby employ the spectral results obtained for the static bifurcation problem dealt with by Ottosen and Runesson (1991). It is also shown that for a broad class of nonassociated plasticity, "flutter" instability cannot occur and we identify the interesting phenomenon of diffuse wave modes.

PLASTICITY FORMULATION

For the sake of simplicity, we shall assume that displacements and strains are small. With σ_{ij} and ε_{ij} being the stress and strain tensor, respectively, and a dot denoting the time derivative, the constitutive relations appropriate for a nonassociated flow rule are

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$$\dot{\sigma}_{ij} = D_{ijkl}\dot{\varepsilon}_{kl} \tag{1}$$

where the bilinear tangent stiffness tensor D_{ijkl} is given by

$$D_{ijkl} = \begin{cases} D^{e}_{ijkl} & \text{(E)} \\ D^{e}_{ijkl} - \dfrac{1}{A} D^{e}_{ijst} g_{st} f_{mn} D^{e}_{mnkl} & \text{(P)} \end{cases} \tag{2}$$

for elastic (E) and plastic (P) loading, respectively. Here D^{e}_{ijkl} denotes the isotropic elastic stiffness tensor, which is assumed to be positive definite, constant and symmetric, i.e. $D^{e}_{ijkl} = D^{e}_{klij}$. Moreover,

$$f_{ij} = \frac{\partial f}{\partial \sigma_{ij}}; \quad g_{ij} = \frac{\partial g}{\partial \sigma_{ij}} \tag{3}$$

where f and g are the yield function and plastic potential, respectively, which depend on the stress state and a set of hardening variables. The positive quantity A is defined by

$$A = H + f_{ij} D^{e}_{ijkl} g_{kl} > 0 \tag{4}$$

where H is the generalized plastic modulus. H is positive, zero or negative for hardening, perfect or softening plasticity, respectively. It appears that D_{ijkl} is symmetric ($D_{ijkl} = D_{klij}$) for associated plasticity, i.e. $f_{ij} = g_{ij}$, whereas D_{ijkl} is nonsymmetric for nonassociated plasticity.

Plastic loading (P) will take place whenever the stress state is on the yield surface and

$$f_{ij} D^{e}_{ijkl} \dot{\varepsilon}_{kl} \geqslant 0. \tag{5}$$

Otherwise, elastic behaviour (E) occurs. Here, we shall only consider the case of plastic loading.

The motion of a so-called singular surface, across which variables may be discontinuous, is the main subject of this study. The jump conditions associated with such discontinuities were already established in the pioneering work of Hadamard (1903). However, it seems appropriate to present a short derivation of the most important jump conditions while only assumptions that are relevant to the present analysis are made.

PRELIMINARIES

Considering the vector $f_i = f_i(x_k)$, where x_k is the position vector, we have

$$\mathrm{d}f_i = \frac{\partial f_i}{\partial x_k} \mathrm{d}x_k. \tag{6}$$

The length of $\mathrm{d}x_k$ is denoted by $|\mathrm{d}x_k| = \mathrm{d}s$, i.e. the unit vector s_k in the direction of $\mathrm{d}x_k$ is given by $s_k = \mathrm{d}x_k/\mathrm{d}s$. From eqn (6) we then obtain

$$\frac{\mathrm{d}f_i}{\mathrm{d}s} = \frac{\partial f_i}{\partial x_k} s_k. \tag{7}$$

If the vector f_i is constant on a surface S, eqn (7) yields:

$$\frac{\mathrm{d}f_i}{\mathrm{d}s} = \frac{\partial f_i}{\partial x_k} t_k = 0 \tag{8}$$

where t_k denotes a unit vector tangential to the surface S. According to eqn (8) the vector $\partial f_1/\partial x_k$ is normal to the arbitrary tangential vector t_k, i.e. we have $\partial f_1/\partial x_k = c_1 n_k$, where n_k is the unit vector normal to the surface S and c_1 is a scalar. The same arguments hold for the vectors $\partial f_2/\partial x_k$ and $\partial f_3/\partial x_k$, i.e. eqn (8) has the solution

$$\frac{\partial f_i}{\partial x_k} = c_i n_k. \tag{9}$$

If, in eqn (7), we let s_k be n_k, then we use the notation $\mathrm{d}s = \mathrm{d}n$ and find with eqn (9) that

$$\frac{\mathrm{d}f_i}{\mathrm{d}n} = c_i, \tag{10}$$

i.e. eqn (9) can be written as

$$\frac{\partial f_i}{\partial x_k} = \frac{\mathrm{d}f_i}{\mathrm{d}n} n_k. \tag{11}$$

If, instead of a vector function f_i, we study a scalar function $f(x_i)$ that is constant on the surface S, then eqn (11) reduces to the familiar expression for the gradient

$$\frac{\partial f}{\partial x_k} = \frac{\mathrm{d}f}{\mathrm{d}n} n_k. \tag{12}$$

Similarly, considering a tensor function $f_{ij}(x_k)$ that is constant on the surface S, we obtain

$$\frac{\partial f_{ij}}{\partial x_k} = \frac{\mathrm{d}f_{ij}}{\mathrm{d}n} n_k. \tag{13}$$

JUMP CONDITIONS

We shall study the motion of a surface S through the body. This surface divides the body so that a quantity—say f_{ij}—has one value f''_{ij} on one side of S and another value f'_{ij} on the other side of S. The difference of f_{ij} across S will, as usual, be denoted by $[f_{ij}]$, i.e.

$$[f_{ij}] = f''_{ij} - f'_{ij}.$$

We here consider a state where the displacement u_i, the velocity $\dot{u}_i$, the displacement gradient $u_{i,j} = \partial u_i/\partial x_j$, the strain ε_{ij} and the stress σ_{ij} are continuous across S, i.e.

$$[u_i] = [\dot{u}_i] = [u_{i,j}] = [\varepsilon_{ij}] = [\sigma_{ij}] = 0 \tag{14}$$

and we shall investigate the possibility that a surface S can exist, across which the velocity gradient $\dot{u}_{i,j}$, the stress rate $\dot{\sigma}_{ij}$ and the acceleration $\ddot{u}_i$ become discontinuous. If such a surface exists then, since $[\dot{u}_i] = [\sigma_{ij}] = 0$ holds on the surface, then we can use eqns (10), (11) and (13) to obtain

$$\frac{\partial[\dot{u}_i]}{\partial x_k} = c_i n_k; \quad c_i = \frac{\mathrm{d}[\dot{u}_i]}{\mathrm{d}n}; \quad \text{i.e. } [\dot{\varepsilon}_{ij}] = \tfrac{1}{2}(c_i n_j + c_j n_i) \tag{15}$$

and

$$\frac{\partial[\sigma_{ij}]}{\partial x_k} = \frac{\mathrm{d}[\sigma_{ij}]}{\mathrm{d}n} n_k. \tag{16}$$

Now let $x_i(q, t)$ be an arbitrary curve on S, where q is a curve parameter and t is time. The velocity of a point on this curve is given by $\dot{x}_i$. The component of $\dot{x}_i$ in the direction of the normal n_i to the surface is by definition the so-called wave speed U, i.e.

$$U = \dot{x}_i n_i. \tag{17}$$

In other words the wave speed is the speed with which S travels in the direction of n_i.

Assume that the vector function $f_i = f_i(x_k, t)$ is always zero on the surface S. Differentiation with respect to time yields

$$\dot{f}_i + \frac{\partial f_i}{\partial x_k}\dot{x}_k = 0$$

where $\dot{f}_i = \partial f_i/\partial t$. Use of eqns (11) and (17) gives

$$\dot{f}_i + U\frac{\mathrm{d}f_i}{\mathrm{d}n} = 0. \tag{18}$$

Likewise, for the tensor function $f_{ij}(x_k, t) = 0$ on the surface S, we obtain

$$\dot{f}_{ij} + \frac{\partial f_{ij}}{\partial x_k}\dot{x}_k = 0$$

and use of eqns (13) and (17) results in

$$\dot{f}_{ij} + U\frac{\mathrm{d}f_{ij}}{\mathrm{d}n} = 0. \tag{19}$$

As $[\dot{u}_i] = [\sigma_{ij}] = 0$ on S, setting $f_i = [\dot{u}_i]$ in eqn (18) and $f_{ij} = [\sigma_{ij}]$ in eqn (19) provides

$$[\ddot{u}_i] + U\frac{\mathrm{d}[\dot{u}_i]}{\mathrm{d}n} = 0 \tag{20a}$$

$$[\dot{\sigma}_{ij}] + U\frac{\mathrm{d}[\sigma_{ij}]}{\mathrm{d}n} = 0. \tag{20b}$$

Multiplying the last equation by n_j and using eqn (16) yields

$$[\dot{\sigma}_{ij}]n_j + U\frac{\partial[\sigma_{ij}]}{\partial x_j} = 0. \tag{21}$$

On each side of the surface S, all variables are continuous and differentiable, i.e. the standard equilibrium conditions hold and we can therefore write

$$\frac{\partial\sigma''_{ij}}{\partial x_j} + b''_i = \rho''\ddot{u}''_i; \quad \frac{\partial\sigma'_{ij}}{\partial x_j} + b'_i = \rho'\ddot{u}'_i.$$

We shall also assume that the body force b_i and mass density ρ are continuous across the surface S, i.e. $b''_i = b'_i = b_i$ and $\rho'' = \rho' = \rho$. Subtraction of the two equilibrium conditions thus gives

$$\frac{\partial[\sigma_{ij}]}{\partial x_j} = \rho[\ddot{u}_i]. \tag{22}$$

GENERAL ACCELERATION WAVE ANALYSIS—PLANE WAVES—STABILITY

We are now in a position to formulate the acceleration wave equations. Use of eqn (22) in (21) results in

$$[\dot{\sigma}_{ij}]n_j + \rho U[\ddot{u}_i] = 0$$

and from eqns (15) and (20a) we have

$$[\ddot{u}_i] = -Uc_i.$$

Combining these equations, we obtain

$$[\dot{\sigma}_{ij}]n_j = \rho U^2 c_i. \tag{23}$$

The stress rates are assumed to correspond to plastic loading, and as the stresses and strains are continuous, the elasto-plastic tangent stiffness tensor D_{ijkl} is also continuous across S, i.e.

$$[\dot{\sigma}_{ij}]n_j = n_j D_{ijkl}[\dot{\varepsilon}_{kl}] = n_j D_{ijkl} n_k c_l \tag{24}$$

where eqn (15) has been used. From eqns (23) and (24), we finally obtain the equations which control acceleration waves

$$Q_{il}c_l = \rho U^2 c_i \tag{25}$$

where the so-called acoustic tensor Q_{il} is given by

$$Q_{il} = n_j D_{ijkl} n_k. \tag{26}$$

In the analysis of static bifurcation Q_{il} is often called the characteristic stiffness tensor. It appears that Q_{il} depends on the material parameters as well as on the direction n_i. The equation system (25) constitutes an eigenvalue problem with ρU^2 being the eigenvalue and c_i the eigenvector and we shall later present analytical solutions for this eigenvalue problem. Since only certain c_i-vectors are eigenvectors, these are said to be polarized and therefore the Q_{il}-tensor is occasionally termed the polarization tensor. If the wave speed U is zero, eqn (25) reduces to the static bifurcation condition considered by Ottosen and Runesson (1991), i.e. the surface S, across which jumps may occur in the stress and strain rates, is then fixed in the body.

Another important phenomenon controlled by eqn (25) is that of propagation of plane waves. Whereas eqn (25) was derived on the basis that a surface S—across which the stress and strain rates as well as the acceleration are discontinuous—travels through the material, we shall now show that the equations for the existence of plane waves are formally the same, even though plane waves do not necessarily involve discontinuities.

A plane wave in direction n_k is defined by

$$u_i = c_i f(n_k x_k \pm Ut) \tag{27}$$

where c_i, n_i and U are constants and f denotes an arbitrary function. If f is twice differentiable, we have

$$\ddot{u}_i = U^2 c_i \frac{\partial^2 f}{\partial(n_k x_k \pm Ut)^2}; \quad u_{k,lj} = c_k n_l n_j \frac{\partial^2 f}{\partial(n_k x_k \pm Ut)^2}. \tag{28}$$

Assume that the material is stressed to a certain state and that this state is in static equilibrium, i.e.

$$\sigma_{ij,j} + \rho b_i = 0. \tag{29}$$

We now investigate the existence of small vibrations about this equilibrium state assuming that the body forces are unchanged. The additional small stresses and displacements caused by the vibrations are denoted by σ_{ij}^* and u_i, respectively. It follows that

$$(\sigma_{ij} + \sigma_{ij}^*)_{,j} + \rho b_i = \rho \ddot{u}_i. \tag{30}$$

Subtraction of eqn (29) from (30) gives

$$\sigma_{ij,j}^* = \rho \ddot{u}_i. \tag{31}$$

Assume now that the material is in an homogeneous state in its original equilibrium configuration. Then the elastic–plastic constitutive tensor is constant throughout the body, i.e. eqn (31) yields

$$D_{ijkl} u_{k,lj} = \rho \ddot{u}_i.$$

By using eqn (28) in this expression, we recover the acceleration wave eqn (25), which demonstrates the fact that even though acceleration waves and plane waves in general are physically distinct phenomena, they are controlled by the same equations. As the investigation of plane waves was based on small vibrations about an already stressed state, this is equivalent to the so-called acoustic approximation in fluid mechanics and for this reason, Q_{il} in eqn (25) is referred to as the acoustic tensor.

If the eigenvalues ρU^2 of eqn (25) are real and positive, both acceleration waves and plane waves exist. Since the amplitude of the function f in eqn (27) is small, u_i will always remain small. This signals a stable situation. If ρU^2 is real but negative, then the corresponding acceleration wave does not exist, but plane waves will still be possible. To see this, we note that any linear combination of solutions of the form (27) is a valid plane wave solution. Suppose that $U^2 = -\alpha^2$, i.e. $U = \pm i\alpha$, where α is positive. Since ρU^2 is real, the corresponding eigenvector c_i is also real and $U = \pm i\alpha$ corresponds to the same eigenvector. Choosing f as a sine function in eqn (27), the following plane wave is possible

$$u_i = c_i[\sin(n_k x_k + i\alpha t) + \sin(n_k x_k - i\alpha t)] = 2c_i \cos(i\alpha t)\sin(n_k x_k).$$

Using Euler's formula, we find that

$$u_i = c_i(e^{\alpha t} + e^{-\alpha t})\sin(n_k x_k). \tag{32}$$

It appears that expression (32), which was derived on condition that ρU^2 is real but negative, provides a solution where the displacement u_i increases with time. This indicates that an arbitrary small disturbance can grow infinitely large with time and we clearly have an unstable situation. As U^2 is negative and as solution (32) for a fixed x_k-vector increases with time without any oscillations, it is common to term the behaviour given by eqn (32) as "divergence" instability in accordance with the terminology in aerodynamics, cf. Rice (1976) and Leipholz (1972).

For linear elasticity as well as associated plasticity, the acoustic tensor Q_{il} in eqn (25) is symmetric, i.e. the eigenvalues are always real. As the elastic constitutive tensor D_{ijkl}^e is assumed to be positive definite, we have $y_i Q_{il}^e y_l = y_i n_j D_{ijkl}^e n_k y_l > 0$, where $Q_{il}^e = n_j D_{ijkl}^e n_k$

is the elastic acoustic tensor and y_i is an arbitrary vector, i.e. Q^e_{il} is also positive definite. For associated plasticity, we will show that the eigenvalues are positive as long as the static bifurcation condition has not been achieved, whereas one eigenvalue becomes negative after static bifurcation has become possible. In this latter case we have "divergence" instability.

For nonassociated plasticity, the acoustic tensor Q_{il} is nonsymmetric, i.e. in principle it is possible to have complex values of ρU^2, i.e. U possess both a real and an imaginary part. Borrowing again the terminology from aerodynamics, we shall refer to this possibility as "flutter" instability, cf. Rice (1976) and Leipholz (1972), since the corresponding u_i-solution for a fixed x_i-vector can be shown to consist of oscillations with increasing amplitude. However, we shall see that for a very broad class of nonassociated plasticity models, "flutter" instability cannot occur. Before this result can be established, we shall derive analytical expressions for the eigenvalues and eigenvectors of eqn (25) that are applicable to general nonassociated plasticity.

EIGENVALUES OF THE ACOUSTIC TENSOR

We shall now determine the eigenvalues μ of the eigenvalue problem (25)

$$Q_{il}c_l = \mu c_i; \quad \mu = \rho U^2 \tag{33}$$

for the case of elastic isotropy and general nonassociated plasticity. In this case the elastic stiffness tensor is given by

$$D^e_{ijkl} = \frac{E}{1+\nu}\left[\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+\frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}\right] \tag{34}$$

where E = Young's modulus and ν = Poisson's ratio ($E > 0$, $-1 < \nu < \frac{1}{2}$). The elastic acoustic tensor Q^e_{il} becomes

$$Q^e_{il} = n_j D^e_{ijkl} n_k = G\left(\frac{1}{1-2\nu}n_i n_l + \delta_{il}\right) \tag{35}$$

where G is the shear modulus. Since Q^e_{il} is positive definite, its inverse P^e_{il} exists and is given by

$$P^e_{ij} = \frac{1}{G}\left[-\frac{1}{2(1-\nu)}n_i n_j + \delta_{ij}\right]; \quad P^e_{ij}Q^e_{il} = \delta_{jl}. \tag{36}$$

It appears that both P^e_{il} and Q^e_{il} are symmetric. From eqns (26), (2) and (35) we obtain

$$Q_{il} = Q^e_{il} - \frac{1}{A}b_i a_l \tag{37}$$

where

$$a_l = f_{mn}D^e_{mnkl}n_k; \quad b_i = n_j D^e_{ijst}g_{st}. \tag{38}$$

With D^e_{ijkl} being isotropic, we obtain

$$a_l = 2Gp_l; \quad b_i = 2Gq_i \tag{39}$$

where

$$p_i = f_{ik}n_k + \frac{v}{1-2v}n_i f_r; \quad f_r = f_{ss} \tag{40}$$

$$q_i = g_{ik}n_k + \frac{v}{1-2v}n_i g_r; \quad g_r = g_{ss}. \tag{41}$$

Combining eqns (37) and (39) gives

$$Q_{il} = Q_{il}^{c} - \frac{4G^2}{A}q_i p_l. \tag{42}$$

Using eqn (35) we may now reformulate the eigenvalue problem (33) as

$$\left[\left(1-\frac{\mu}{G}\right)\delta_{il} + \frac{1}{1-2v}n_i n_l - \frac{4G}{A}q_i p_l\right]c_l = 0. \tag{43}$$

In order to facilitate the eigenvalue analysis, we shall first obtain some useful preliminary results. Consider the eigenvalue problem

$$P_{il}^{c}z_l = \lambda^* z_i$$

which due to eqn (36) can be written as

$$\left[(1-\lambda^* G)\delta_{il} - \frac{1}{2(1-v)}n_i n_l\right]z_l = 0. \tag{44}$$

Assuming $\lambda^* = 1/G$, then eqn (44) reduces to $n_i n_l z_l = 0$. In the coefficient matrix $n_i n_l$ all rows are proportional, which proves that $\lambda^* = 1/G$ is an eigenvalue with a multiplicity of two. We therefore obtain

$$\lambda_1^* = \lambda_2^* = \frac{1}{G}; \quad \lambda_3^* = \frac{1-2v}{2G(1-v)} = \frac{1}{M} \tag{45}$$

where the last eigenvalue λ_3^* is obtained from the invariant condition $P_{ii}^{c} = \lambda_1^* + \lambda_2^* + \lambda_3^*$. It appears that the parameter M is the elastic "constrained modulus" pertinent to the case of uniaxial strain.

Next, consider the eigenvalue problem

$$B_{il}y_l = \lambda y_i \tag{46}$$

where

$$B_{il} = P_{ij}^{c}Q_{jl} = \delta_{il} - \frac{1}{A}P_{ij}^{c}b_j a_l \tag{47}$$

and where use has been made of eqns (36) and (37). It was shown by Ottosen and Runesson (1991) that the eigenvalues are given as

$$\lambda_1 = \lambda_2 = 1; \quad \lambda_3 = 1 - \frac{1}{A}b_i P_{il}^{c}a_l.$$

In the present case where elastic isotropy is assumed, we find with eqns (36) and (39) that

$$\lambda_1 = \lambda_2 = 1; \quad \lambda_3 = 1 + \frac{2G}{A}\left(\frac{1}{1-v} n_i q_i n_j p_j - 2p_i q_i\right). \tag{48}$$

With these preliminary results we are now in the position to determine the eigenvalues μ of eqn (43). It appears that one eigenvalue is $\mu_1 = G$. This is shown by introducing $\mu_1 = G$ into eqn (43) while observing that one of the rows in the coefficient matrix to c_i can be expressed as a linear combination of the remaining two rows. Hence, it follows that the coefficient matrix is singular.

To determine the remaining eigenvalues μ_2 and μ_3, we first use the invariant property

$$\mu_2 + \mu_3 = Q_{ii} - \mu_1 = G + M - \frac{4G^2}{A} q_i p_i. \tag{49}$$

Moreover, from eqn (47) we obtain

$$\det B_{il} = \det P^e_{ij} \det Q_{jl}$$

which gives

$$\mu_1 \mu_2 \mu_3 = \frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_1^* \lambda_2^* \lambda_3^*} \tag{50}$$

where λ_i^* and λ_i were given by eqns (45) and (48), respectively. With $\mu_1 = G$ we may thus solve for μ_2 and μ_3 from eqns (49) and (50) to obtain

$$\mu_1 = G \tag{51}$$

$$\left.\begin{matrix}\mu_2\\ \mu_3\end{matrix}\right\} = \frac{1}{2}\left[G + M - \frac{4G^2}{A} q_i p_i \pm \sqrt{\left(G + M - \frac{4G^2}{A} q_i p_i\right)^2 - 4\lambda_3 GM}\right]. \tag{52}$$

Expressions (51) and (52) provide closed-form solutions of the eigenvalue problem (33). Except for the assumption of isotropic elasticity, these eigenvalues apply to general non-associated plasticity. Let us now evaluate these values in more detail.

For linear elasticity we have $A \to \infty$ according to eqn (4), and we obtain from eqns (51) and (52) the familiar expressions

$$\mu_1 = \mu_3 = G; \quad \mu_2 = M \tag{53}$$

which are independent of the direction n_i.

In general, we have $\mu_2 > \mu_3$ and the usual static bifurcation condition ($U^2 = 0$) emerges when $\mu_3 = 0$, which according to eqn (52) requires $\lambda_3 = 0$ in accordance with Ottosen and Runesson (1991). This corresponds to the critical hardening modulus $H = H_b$. It is of interest that $\mu_1 = G$ is constant always, whereas μ_2 and μ_3 change with the loading as well as with the direction n_i. When $H < H_b$, which implies that $\lambda_3 < 0$, it appears from eqn (52) that

$$\mu_2 > 0; \quad \mu_3 < 0 \tag{54}$$

i.e. there is no acceleration wave speed for the solution corresponding to μ_3. According to the previous discussion, this situation is referred to as "divergence" instability. For associated plasticity the acoustic tensor is symmetric, i.e. its eigenvalues are always real, but we shall later prove that the same applies to a very general class of nonassociated plasticity. Before this topic is investigated, it is of interest to determine the eigenvectors c_i of the eigenvalue problem (33).

EIGENVECTORS OF THE ACOUSTIC TENSOR

Let us now determine the eigenvectors of the acoustic problem (33). When $\mu = \mu_1 = G$, eqn (43) reduces to

$$\left(\frac{1}{1-2\nu} n_i n_j - \frac{4G}{A} q_i p_j\right) c_j^{(1)} = 0. \tag{55}$$

i.e., the eigenvector $c_i^{(1)}$ corresponding to $\mu = \mu_1$ is orthogonal to n_i as well as to p_i, i.e.

$$c_i^{(1)} \text{ orthogonal to } n_i \text{ and } p_i. \tag{56}$$

It is of interest to investigate the loading criterion (5) for the strain rate tensor $[\dot{\varepsilon}_{ij}] = (c_i n_j + c_j n_i)/2$ in the case where c_i is the eigenvector $c_i^{(1)}$. We obtain

$$f_{ij} D^e_{ijkl}[\dot{\varepsilon}_{kl}] = f_{ij} D^e_{ijkl} n_k c_l^{(1)} = 2G p_l c_l^{(1)} = 0.$$

This result follows from eqns (38) and (39) as well as the orthogonality of p_i and $c_i^{(1)}$ apparent from eqn (56). Thus, the strain rate tensor resulting from the eigenvector $c_i^{(1)}$ corresponds in fact to neutral loading and this result also substantiates the finding that $\mu_1 = G$.

When $\mu = \mu_2$ or $\mu = \mu_3$ (i.e. $\mu \neq G$), we obtain from eqn (43) that the corresponding eigenvectors, denoted by $c_i^{(2)}$ and $c_i^{(3)}$, must be given by

$$c_i^{(k)} = n_i + \alpha^{(k)} q_i; \quad k = 2, 3 \tag{57}$$

where $\alpha^{(k)}$ are parameters to be determined. Inserting eqn (57) into (43) yields the conditions

$$\alpha^{(k)} = \frac{1}{q_i n_i}\left[\left(\frac{\mu_k}{G} - 1\right)(1-2\nu) - 1\right] \tag{58}$$

$$\alpha^{(k)} = \frac{\frac{4G}{A} n_i p_i}{1 - \frac{\mu_k}{G} - \frac{4G}{A} q_i p_i} \tag{59}$$

for $k = 2$ and 3. The equivalence of these two conditions results in an expression, which, after some algebra, is easily shown to be identical with eqn (52).

Hill (1962) investigated the case of large strains in conjunction with associated plasticity and gave expressions from which the eigenvalues and eigenvectors can be determined. To compare the present results with Hill's expressions in the case of small strains, we can insert μ_k as given by eqn (52) into the expression for $\alpha^{(k)}$ as given by eqn (58). In the case of associated plasticity we obtain, after some algebraic manipulations, the same expressions for the eigenvectors as provided by Hill, cf. eqns (43)–(45) in Hill (1962). Also the expressions for the eigenvalues provided by eqns (51) and (52) reduce for associated plasticity to those given by Hill, cf. eqns (43) and (45a) in Hill (1962). In fact, eqn (45a) in Hill (1962) emerges from our eqn (58).

Clearly, for $c_i^{(2)}$ and $c_i^{(3)}$ to be orthogonal to $c_i^{(1)}$, they have to lie in the plane spanned by n_i and p_i, since $c_i^{(1)}$ is orthogonal to this plane, cf. eqn (56). However, in the case of general nonassociated plasticity $q_i \neq p_i$ and, furthermore, q_i is not spanned by n_i and p_i. This implies that $c_i^{(2)} c_i^{(1)} \neq 0$ and $c_i^{(3)} c_i^{(1)} \neq 0$. It then follows that $c_i^{(2)}$ and $c_i^{(3)}$ are orthogonal to $c_i^{(1)}$ only in the case of associated plasticity where $q_i = p_i$.

It is also concluded that $c_i^{(2)}$ and $c_i^{(3)}$ are mutually nonorthogonal in the general case, which can be shown by considering the scalar product $c_i^{(2)} c_i^{(3)}$. From eqn (57) we obtain

$$c_i^{(2)}c_i^{(3)} = 1+(\alpha^{(2)}+\alpha^{(3)})n_i q_i+\alpha^{(2)}\alpha^{(3)}q_i q_i. \tag{60}$$

Using eqn (59) for example, we obtain, after some algebraic manipulation,

$$\alpha^{(2)}\alpha^{(3)} = (1-2\nu)\frac{4G}{A}\frac{n_i p_i}{n_i q_i}$$

$$\alpha^{(2)}+\alpha^{(3)} = -\frac{1}{n_i q_i}\left[1+(1-2\nu)\frac{4G}{A}q_k p_k\right]$$

which inserted into eqn (60) gives

$$c_i^{(2)}c_i^{(3)} = (1-2\nu)\frac{4G}{A}\left(\frac{n_i p_i}{n_i q_i}q_k q_k - q_i p_i\right). \tag{61}$$

It follows that $c_i^{(2)}$ and $c_i^{(3)}$ are orthogonal only when $p_i = q_i$, i.e. in the case of associated plasticity. Consequently, only associated plasticity implies that all the eigenvectors are orthogonal, whereas all eigenvectors become nonorthogonal whenever nonassociated plasticity is employed. That associated plasticity implies orthogonal eigenvector follows, of course, trivially from the symmetry of the acoustic tensor.

Finally, let us consider the limiting case of linear elasticity defined by $A \to \infty$, where the eigenvalues are given by eqn (53). For $\mu_1 = \mu_3 = G$, eqn (55) with $A \to \infty$ holds for both $c_i^{(1)}$ and $c_i^{(3)}$, i.e. both $c_i^{(1)}$ and $c_i^{(3)}$ are orthogonal to n_i and they can be taken as mutually orthogonal. For $\mu_2 = M$ both eqn (58) and (59) imply that $\alpha^{(2)} = 0$, i.e. eqn (57) gives that $c_i^{(2)} = n_i$. We have thus rediscovered that the longitudinal (dilatational) waves corresponding to $\mu_2 = M$ travel in the direction of n_i, whereas the transverse (shear) waves corresponding to $\mu_1 = \mu_3 = G$ travel transverse to the n_i-direction (as the notation indicates).

We may summarize by stating that for linear elasticity the two transverse waves are orthogonal to the longitudinal wave and the transverse waves can be chosen as mutually orthogonal. For associated plasticity, all the eigenvectors are always orthogonal, whereas nonassociated plasticity implies that all eigenvectors become nonorthogonal.

WHEN ARE THE EIGENVALUES REAL?

We shall now prove that for a very broad class of nonassociated plasticity, the eigenvalues are always real, implying that "divergence" instability can occur, while the phenomenon of "flutter" instability cannot. Divergence instability occurs if the discriminant D in eqn (52)

$$D = \left(G+M-\frac{4G^2}{A}q_i p_i\right)^2 - 4\lambda_3 GM \tag{62}$$

is non-negative always. We observe that when $\lambda_3 \leqslant 0$ then $D \geqslant 0$ holds always, i.e. the possibility for complex eigenvalues expressed through $D < 0$ exists only when $\lambda_3 > 0$, which corresponds to the regime before static bifurcation becomes possible. On introducing the notation

$$N_1 = \frac{2G}{A} q_i p_i \tag{63}$$

$$N_2 = \frac{2G}{A} q_i n_i p_j n_j \tag{64}$$

we can rewrite λ_3 given by eqn (48) as

$$\lambda_3 = 1 + \frac{N_2}{1-\nu} - 2N_1. \tag{65}$$

With eqns (63) and (65), eqn (62) takes the form

$$D = 4G^2\left[\left(N_1 - \frac{M-G}{2G}\right)^2 - \frac{2}{1-2\nu}(N_2 - N_1)\right]. \tag{66}$$

Let us also introduce the definitions

$$q_i p_i = |q|\,|p| \cos\theta; \quad q_i n_i = |q| \cos\theta_q; \quad p_i n_i = |p| \cos\theta_p. \tag{67}$$

Then we can write

$$N_2 - N_1 = \frac{2G}{A} |p|\,|q|(\cos\theta_p \cos\theta_q - \cos\theta). \tag{68}$$

For associated plasticity we have $p_i = q_i$ implying that $\theta = 0$ and $\theta_q = \theta_p$, i.e. eqn (68) becomes

$$N_2 - N_1 = \frac{2G}{A} |p|^2(\cos^2\theta_p - 1) \leqslant 0$$

where the inequality is valid since $\cos^2\theta_p - 1 \leqslant 0$. In this case it appears from eqn (66) that $D \geqslant 0$, which proves the already known fact that for associated plasticity the eigenvalues are always real.

With this observation, it becomes natural to investigate the sign of $N_2 - N_1$ for non-associated plasticity as well, and we conclude that the eigenvalues are always real whenever

$$\varphi = q_i n_i p_j n_j - q_i p_i \leqslant 0 \tag{69}$$

holds, which implies that

$$N_2 - N_1 = \frac{2G}{A}\varphi \leqslant 0. \tag{70}$$

To prove this inequality for nonassociated plasticity, we shall make two minor assumptions, which are valid for all practical purposes.

The first assumption is that $f_{ij} = \partial f/\partial\sigma_{ij}$ and $g_{ij} = \partial g/\partial\sigma_{ij}$ possess the same principal directions. This assumption is valid for general mixed isotropic–kinematic hardening, where $f = f(\sigma_{ij} - \alpha_{ij}, \kappa_\alpha)$ and $g = g(\sigma_{ij} - \alpha_{ij}, \kappa_\alpha)$ and f and g are isotropic functions of the argument $\sigma_{ij} - \alpha_{ij}$. In these expressions, α_{ij} denotes a tensorial hardening parameter ("backstress") and κ_α ($\alpha = 1, 2, \ldots$) are scalar hardening parameters. The principal directions of f_{ij} and g_{ij} then coincide with those of the tensor $\sigma_{ij} - \alpha_{ij}$. We may now conveniently choose the coordinate system colinear with the principal directions of f_{ij} and g_{ij} to obtain

$$f_{ij} = \begin{pmatrix} f_1 & 0 & 0 \\ 0 & f_2 & 0 \\ 0 & 0 & f_3 \end{pmatrix}; \quad g_{ij} = \begin{pmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{pmatrix} \tag{71}$$

where f_1, f_2, f_3 and g_1, g_2, g_3 denote the principal values of f_{ij} and g_{ij}, respectively. Without loss of generality we can take

$$f_1 \geqslant f_2 \geqslant f_3. \tag{72}$$

The second assumption is that eqn (72) implies

$$g_1 \geqslant g_2 \geqslant g_3. \tag{73}$$

Even though we have not been able to prove that eqns (71) and (72) imply (73), assumption (73) still comprises a very general class of nonassociated plasticity behaviour. In fact, we have not been able to identify any existing constitutive model, which satisfies eqn (71), but for which eqn (72) does not infer (73). A trivial example, which satisfies eqns (71)–(73), is associated deviatoric behaviour and nonassociated volumetric behaviour; this case includes the classic nonassociated Drucker–Prager model.

From eqns (40) and (41) we now obtain

$$p_i = \begin{bmatrix} (f_1 + \gamma f_v)n_1 \\ (f_2 + \gamma f_v)n_2 \\ (f_3 + \gamma f_v)n_3 \end{bmatrix}; \quad q_i = \begin{bmatrix} (g_1 + \gamma g_v)n_1 \\ (g_2 + \gamma g_v)n_2 \\ (g_3 + \gamma g_v)n_3 \end{bmatrix} \quad \gamma = \frac{v}{1-2v}. \tag{74}$$

This yields

$$R = p_i n_i = f_1 n_1^2 + f_2 n_2^2 + f_3 n_3^2 + \gamma f_v \tag{75}$$

$$S = q_i n_i = g_1 n_1^2 + g_2 n_2^2 + g_3 n_3^2 + \gamma f_v \tag{76}$$

$$p_i q_i = K_1 n_1^2 + K_2 n_2^2 + K_3 n_3^2 \tag{77}$$

where

$$K_1 = (f_1 + \gamma f_v)(g_1 + \gamma g_v); \quad K_2 = (f_2 + \gamma f_v)(g_2 + \gamma g_v); \quad K_3 = (f_3 + \gamma f_v)(g_3 + \gamma g_v). \tag{78}$$

Inserting eqns (75)–(77) in eqn (69) results in

$$\varphi = RS - K_1 n_1^2 - K_2 n_2^2 - K_3 n_3^2. \tag{79}$$

As we want to prove inequality (69), it is natural to determine the extrema of φ with respect to variations of the n_i-vector. Details of this analysis are given in the Appendix, where it is proven that $\varphi \leqslant 0$ always holds. Therefore, according to eqn (70) it follows that $N_2 - N_1 \leqslant 0$, and from eqn (66) it follows that the discriminant D appearing in eqn (54) is always non-negative, i.e. the eigenvalues of the acoustic tensor Q_{ij} are always real. Consequently, the so-called "flutter" instability discussed in the literature, e.g. Rice (1976), cannot occur.

DIFFUSE WAVE MODES

We shall now prove that some rather dramatic wave motions can occur, which we shall term diffuse wave motions. This phenomenon is related to the situations when $\varphi = 0$. From the Appendix, the pertinent situations can be summarized as shown in Table 1. It appears that for general stress states, where $f_1 \geqslant f_2 \geqslant f_3$ and $g_1 \geqslant g_2 \geqslant g_3$, we have three different choices of n_i for which $\varphi = 0$. For special stress states, other n_i-vectors also imply $\varphi = 0$.

According to eqn (70), the situation $\varphi = 0$ implies that $N_2 - N_1 = 0$ and from eqn (66) it then follows that the discriminant D is given by

$$D = \frac{4G^2}{A}\left(N_1 - \frac{M-G}{2G}\right)^2. \tag{80}$$

A very interesting situation can now arise. Suppose that $D = 0$, i.e.

$$N_1 = \frac{M-G}{2G} = \frac{1}{2(1-2\nu)}. \tag{81}$$

With $D = 0$, it follows from eqns (52), (62) and (63) that

$$\mu_2 = \mu_3 = \tfrac{1}{2}(G + M - 2GN_1) = G$$

where eqn (81) has been used. Therefore, when eqn (81) is fulfilled and the n_i-vector is chosen appropriately, we obtain the interesting situation that all eigenvalues are equal and are given by

$$\mu_1 = \mu_2 = \mu_3 = G, \tag{82}$$

i.e. all waves travel with the same wave speed. In accordance with the definition of N_1 given by eqn (63), the fulfilment of eqn (81) requires a specific value of A, whereby it will be recalled that q_i and p_i are fixed for the chosen value of the n_i-vector. This value of A, in turn, implies a specific value of the hardening modulus H. To evaluate whether this value is physically acceptable, we evaluate λ_3 as given by eqn (65), which can be rewritten as

$$\lambda_3 = 1 + \frac{N_2 - N_1}{1-\nu} - \frac{1-2\nu}{1-\nu}N_1.$$

In the situation under consideration, $\varphi = 0$ holds and since this implies $N_1 = N_2$, where N_1 is given by eqn (81), we obtain

Table 1. Situations where $\varphi = 0$; $f_1 \geqslant f_2 \geqslant f_3$ and $g_1 \geqslant g_2 \geqslant g_3$

Stress state	n_i-vector
$f_1 \geqslant f_2 \geqslant f_3$; $g_1 \geqslant g_2 \geqslant g_3$	$n_1^2 = 1$, $n_2 = n_3 = 0$ $n_1 = 0$, $n_2^2 = 1$, $n_3 = 0$ $n_1 = n_2 = 0$, $n_3^2 = 1$
$f_1 = f_2 > f_3$ and/or $g_1 = g_2 > g_3$	$n_1^2 + n_2^2 = 1$, $n_3^2 = 0$
$f_1 > f_2 = f_3$ and/or $g_1 > g_2 = g_3$	$n_1^2 = 0$, $n_2^2 + n_3^2 = 1$
$f_1 = f_2 = f_3$ and/or $g_1 = g_2 = g_3$	$n_1^2 + n_2^2 + n_3^2 = 1$

$$0 < \lambda_3 = \frac{1-2\nu}{2(1-\nu)} \leqslant \frac{1}{2}. \tag{83}$$

Recalling that $\lambda_3 = 1$ for purely elastic behaviour and $\lambda_3 = 0$ when static bifurcation becomes possible, we conclude that the positive λ_3-value given above is fully acceptable since it corresponds to the regime before static bifurcation becomes possible in the pertinent n_i-direction.

The situation for which eqn (81) is fulfilled has some further dramatic consequences for the eigenvectors. To show this, we have to determine the values of $\cos\theta_p$ and $\cos\theta_q$. As $N_1 = N_2$, we obtain from eqn (68)

$$\cos\theta = \cos\theta_p \cos\theta_q \tag{84}$$

and from eqns (63), (67) and (81) we conclude that

$$\cos\theta > 0. \tag{85}$$

First consider associated behaviour, i.e. $\cos\theta = 1$, for which eqns (84) and (85) yield

$$\cos\theta_p = \cos\theta_q = 1 \quad \text{or} \quad \cos\theta_p = \cos\theta_q = -1. \tag{86}$$

We shall now prove that relations (86) also hold for nonassociated behaviour. Equation (74) results in

$$|p|^2 = (f_1+\gamma f_e)^2 n_1^2 + (f_2+\gamma f_e)^2 n_2^2 + (f_3+\gamma f_e)^2 n_3^2 \tag{87}$$

$$|q|^2 = (g_1+\gamma g_e)^2 n_1^2 + (g_2+\gamma g_e)^2 n_2^2 + (g_3+\gamma g_e)^2 n_3^2 \tag{88}$$

whereas expressions for $p_i n_i = |p|\cos\theta_p$ and $q_i n_i = |p|\cos\theta_q$ are given by eqns (75) and (76). Referring to Table 1 and considering the situation where $f_1 \geqslant f_2 \geqslant f_3$; $g_1 \geqslant g_2 \geqslant g_3$ and $n_1^2 = 1$, $n_2 = n_3 = 0$ we obtain

$$|p|^2 = (f_1+\gamma f_e)^2; \quad |p|\cos\theta_q = f_1+\gamma f_e$$

and

$$|q|^2 = (g_1+\gamma g_e)^2; \quad |q|\cos\theta_q = g_1+\gamma g_e$$

which imply eqn (86). The same result applies to all the cases shown in Table 1 provided that the "and/or" condition is replaced by an "and" condition.

From eqn (86) and the definitions $p_i n_i = |p|\cos\theta_p$ and $q_i n_i = |q|\cos\theta_q$ it follows that either

$$p_i = |p| n_i; \quad q_i = |q| n_i \tag{89}$$

or

$$p_i = -|p| n_i; \quad q_i = -|q| n_i. \tag{90}$$

For the case considered, we recall from eqn (82) that $\mu_1 = \mu_2 = \mu_3 = G$. However, when $\mu = G$ holds, eqn (55) defines the eigenvector c_l, i.e. we have

$$\left(\frac{1}{1-2\nu} n_i n_l - \frac{4G}{A} q_i p_l\right) c_l = 0$$

which due to eqns (89) and (90) takes the form

$$\left(\frac{1}{1-2\nu}-\frac{4G}{A}|p||q|\right)n_i n_j c_j = 0. \quad (91)$$

Due to eqns (84) and (86) the relation $\cos\theta = 1$ holds. Therefore, using eqns (63), (67) and (81) we obtain

$$N_1 = \frac{2G}{A}|q||p| = \frac{1}{2(1-2\nu)}. \quad (92)$$

When this relation is inserted into eqn (91) it implies that the coefficient matrix to c_j vanishes, i.e. eqn (91) is satisfied for arbitrary c_j-vectors.

For associated and nonassociated plasticity, we thus reach the interesting conclusion that a certain value of the hardening modulus exists for which specific choices of the n_i-vector will always imply that all wave speeds are identical and equal to the elastic shear (distortion) wave speed, i.e. $\rho U_{(1)}^2 = \rho U_{(2)}^2 = \rho U_{(3)}^2 = G$, and the corresponding eigenvectors are arbitrary. This value of the hardening modulus corresponds to the regime before static bifurcation becomes possible in the pertinent n_i-direction. In this situation the acoustic (or polarization) tensor therefore loses its ability to provide distinct—polarized—eigenvectors and the wave modes might be termed diffuse. This situation is most remarkable and may have significant ramifications for the interpretation of seismic waves and—in particular—waves from underground explosions, where the material is highly stressed. Moreover, the existence of diffuse wave modes also calls for special precautions in numerical modelling.

EXAMPLES OF MATERIAL BEHAVIOUR

Let us now illustrate some of the findings by considering the two simple plasticity formulations of von Mises and Rankine.

von Mises criterion

An isotropic hardening von Mises model is defined by

$$f = g = \sqrt{3J_2} - \kappa = 0 \quad (93)$$

where the invariant J_2 is given by $J_2 = s_{ij}s_{ij}/2$ with s_{ij} being the deviatoric stress tensor; κ denotes a hardening parameter. Since we assume associated plasticity $p_i = q_i$ holds, and choosing the coordinate sytsem colinear with the principal directions of the stress tensor we obtain from eqn (40) that

$$n_i p_i = \frac{3}{2\sqrt{3J_2}}(n_1^2 s_1 + n_2^2 s_2 + n_3^2 s_3); \quad p_i q_i = \frac{3}{4J_2}(n_1^2 s_1^2 + n_2^2 s_2^2 + n_3^2 s_3^2). \quad (94)$$

For simplicity, we shall in the following assume a uniaxial stress state given by the tensile stress σ_1. It follows that $\sqrt{3J_2} = \sigma_1$, and eliminating n_3^2 through the constraint relation $n_3^2 = 1 - n_1^2 - n_2^2$, we obtain from eqn (94)

$$n_i p_i = \tfrac{1}{2}(3n_1^2 - 1); \quad p_i q_i = \tfrac{1}{4}(3n_1^2 + 1). \quad (95)$$

Moreover, from eqns (4) and (48) it is simple to demonstrate that

$$A = H + 3G; \quad \lambda_3 = 1 + \frac{G}{2A}\left[\frac{1}{1-\nu}(3n_1^2 - 1)^2 - 2(3n_1^2 + 1)\right]. \quad (96)$$

From eqn (45) it follows that

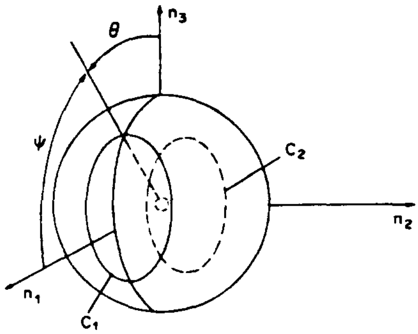

Fig. 1. Unit sphere for the n_i-vector.

$$M = \frac{2G(1-\nu)}{1-2\nu}; \quad M-G = \frac{G}{1-2\nu}; \quad M+G = \frac{G(3-4\nu)}{1-2\nu}. \tag{97}$$

Use of eqns (95)–(97) in eqns (51) and (52) then provides the following expressions for the eigenvalues:

$$\frac{\mu_1}{G} = 1 \tag{98}$$

$$\left.\begin{array}{c}\dfrac{\mu_2}{G}\\[2ex] \dfrac{\mu_3}{G}\end{array}\right\} = \frac{1}{2}\left\{\frac{3-4\nu}{1-2\nu} - \frac{3n_1^2+1}{3+(H/G)} \pm \sqrt{\left[\frac{1}{1-2\nu}+\frac{3n_1^2+1}{3+(H/G)}\right]^2 - \frac{4(3n_1^2-1)^2}{(3+(H/G))(1-2\nu)}}\right\}. \tag{99}$$

It appears that the normalized eigenvalues μ_2/G and μ_3/G depend only on Poisson's ratio ν, the value of n_1^2 and on the ratio H/G. For given values of ν and H/G, it may therefore be of interest to evaluate how μ_2/G and μ_3/G vary with the value of n_1^2. Since only n_1^2 enters the expression the eigenvalues are independent of the specific values of n_2 and n_3 except that the constraint relation $n_2^2+n_3^2 = 1-n_1^2$ always holds. According to Fig. 1, showing the unit sphere at which the n_i-vector is located, this implies that along the two circles C_1 and C_2 given by n_1^2 = constant, we have the same eigenvalues μ_2 and μ_3. We therefore choose to illustrate the variation of μ_2 and μ_3 with the direction of the n_i-vector by considering only the n_1n_3-plane shown in Fig. 2. Alternatively, the n_3-value in Fig. 2 can be interpreted as the value $\pm\sqrt{n_2^2+n_3^2}$. It follows that $\cos\psi = n_1$ and $\theta = (\pi/2)-\psi$, where the angle θ shown in Fig. 1 corresponds to the angle defined in Ottosen and Runesson (1991).

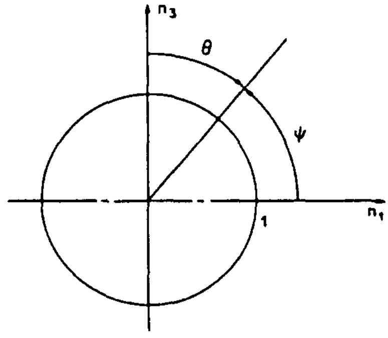

Fig. 2. Plane illustration of variation of n_i.

With these preliminary remarks and recalling that $\mu_1/G = 1$ always holds, we may visualize the magnitude of the normalized eigenvalues μ_2/G and μ_3/G given by eqn (99) in the form of polar diagrams, where the polar angle ψ shown in Fig. 2 is determined by $\cos\psi = n_1$. The results for $v = 0.3$ are shown in Fig. 3 for different values of H/G.

Figure 3(a) shows the linear elastic behaviour, where μ_2 and μ_3 do not depend on the direction of the n_i-vector. Figure 3(b) shows that plasticity introduces a directional dependence, and in accordance with eqn (99) we observe the symmetry about both axes. Figure 3(c) finally illustrates the situation where the eigenvalue μ_3 becomes zero for one direction, i.e. the situation has just been reached where static bifurcation becomes possible. This is equivalent to the condition that $\lambda_3 = 0$ and from eqn (96) we obtain the bifurcation direction

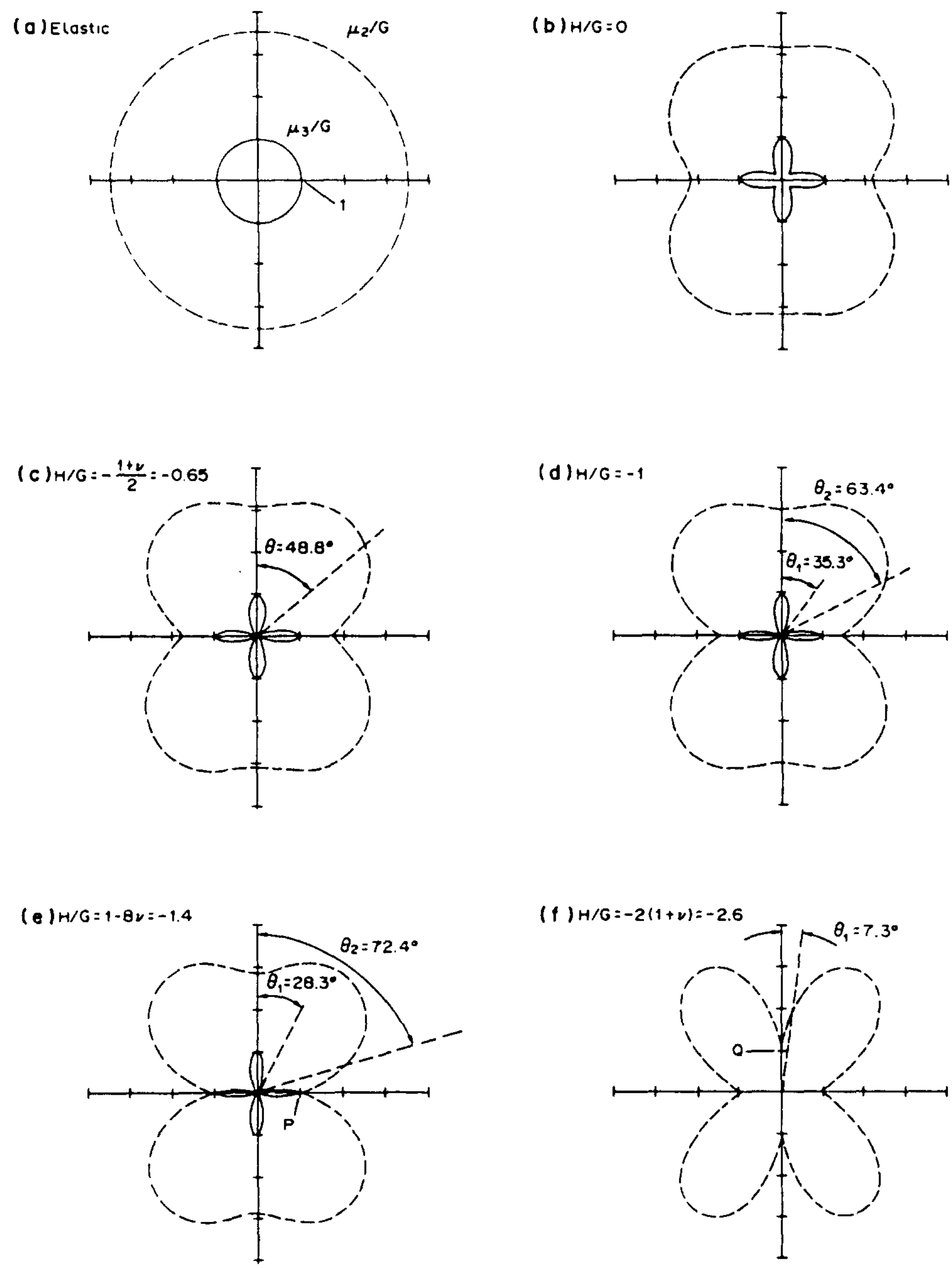

Fig. 3. von Mises plasticity; uniaxial tensile stress; $v = 0.3$. Variation of normalized eigenvalues μ_2/G (— — —) and μ_3/G (———) with the n_1-value.

$$n_1^2 = \frac{1}{3}\left[2-\nu\pm\sqrt{-(1-\nu)\left(1+\nu+2\frac{H}{G}\right)}\right]. \tag{100}$$

The first moment, when this relation becomes possible, occurs when the discriminant is zero which provides

$$\frac{H}{G} = -\frac{1+\nu}{2}; \quad n_1^2 = \frac{2-\nu}{3}. \tag{101}$$

Choosing $n_2 = 0$ we have $n_3^2 = 1-n_1^2 = (1+\nu)/3$, i.e. the angle θ shown in Figs 1 and 2 is determined from $\tan^2\theta = n_1^2/n_3^2 = (2-\nu)/(1+\nu)$ in accordance with the result given by Ottosen and Runesson (1991). For $\nu = 0.3$, the value $\theta \simeq 48.8^\circ$ is obtained, cf. Fig. 3(c). Moreover, from eqn (101) we obtain that the critical hardening modulus, for which static bifurcation becomes possible, is given by $H = -E/4$. In Ottosen and Runesson (1991) the critical hardening modulus was found to be $H = -E/12$; however, since the yield criterion in Ottosen and Runesson (1991) was stated as $f = \sqrt{J_2}-\kappa = 0$, which differs from eqn (93), the value of H in Ottosen and Runesson (1991) is one third of the value of H considered here, i.e. the two expressions for the critical hardening modulus coincide.

In Fig. 3(d) the ratio H/G is chosen as $H/G = -1$, i.e. the softening is more pronounced than that for which static bifurcation first becomes possible. Static bifurcation is still possible since it can occur when $\lambda_3 = 0$, which implies eqn (100). Therefore, for the given ratio, H/G, eqn (100) and the relation $\sin\theta = n_1$ determine two θ angles for which static bifurcation is possible. In the present case we obtain $\theta_1 = 35.3^\circ$ and $\theta_2 = 63.4^\circ$ as shown in Fig. 3(d). Between these angles we have $\lambda_3 < 0$ implying $\mu_3 < 0$, which, in turn, implies a state of divergence instability. These negative values of μ_3 are not shown in Fig. 3(d).

In Fig. 3(e) static bifurcation is possible for $\theta_1 = 28.3^\circ$ and $\theta_2 = 72.4^\circ$ and a state of divergence instability exists between these angles; again the negative values of μ_3 in this region are not plotted. Another interesting phenomenon appears from Fig. 3(e), namely that of diffuse wave motions. It appears that at point P we have $\mu_2/G = \mu_3/G = 1$ (and of course $\mu_1/G = 1$). The point P corresponds to $n_1^2 = 1$. According to eqns (83) and (96) we obtain for diffuse wave modes

$$\lambda_3 = \frac{1-2\nu}{2(1-\nu)} = 1+\frac{1}{2\left(3+\frac{H}{G}\right)}\left[\frac{1}{1-\nu}(3n_1^2-1)^2-2(3n_1^2+1)\right] \tag{102}$$

which, as expected, is fulfilled for $n_1^2 = 1$ and the value $H/G = 1-8\nu$ that was adopted in Fig. 3(e).

In accordance with Table 1 and since $f_1 > f_2 = f_3$ for uniaxial tension, a diffuse wave motion should exist also for $n_1^2 = 0$. This situation is shown in Fig. 3(f) at the point Q where $n_1^2 = 0$ holds. As expected, the value $H/G = -2(1+\nu)$ adopted in Fig. 3(f) and $n_1^2 = 0$ satisfy eqn (102). In Fig. 3(f) it is also of importance to note that the characteristic four-leaf clover-shaped graphs of the variation of μ_3/G have given way to slender two-leaf shape. This is a result of the angle θ_2 being extended to 90°, and in this case only the minus sign in front of the square root in eqn (100) provides a value of n_1^2 in the range $0 \leqslant n_1^2 \leqslant 1$. In the present situation we obtain $\theta_1 = 7.3^\circ$. It is of interest that the value of the hardening modulus used in Fig. 3(f) can be written as $H = -E$ and this corresponds to a softening branch having a vertical slope, i.e. a completely brittle behaviour. However, even in this extreme case the condition $A = H+3G = G(1-2\nu) > 0$ is fulfilled.

We finally observe from Figs 3(a)–3(e) that $\mu_3/G = 1$ holds for $n_1^2 = 0$ or $n_1^2 = 1$. It is easily shown from eqn (99) that this is a general result, which holds as long as $H/G \geqslant 1-8\nu$. From eqn (99) it also follows for $-2(1+\nu) \leqslant H/G < 1-8\nu$ that $n_1^2 = 0$ still implies $\mu_3/G = 1$, whereas $n_1^2 = 1$ now implies $\mu_2/G = 1$, cf. Fig. 3(f). This change of behaviour explains the change of the four-leaf clover shape of Fig. 3(e) to the two-leaf shape of Fig.

3(f). For the remaining range defined by $-3 < H/G < -2(1+\nu)$ it is easily shown from eqn (99) that $n_1^2 = 0$, or $n_1^2 = 1$, always implies $\mu_2/G = 1$.

Rankine criterion

As a material model representative for the modelling of cracks in cementitious material we now consider the associated maximum tension cut-off criterion, i.e.

$$f = g = \sigma_1 - \sigma_t = 0 \tag{103}$$

where $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$ are the principal stresses (tension positive) and $\sigma_t > 0$ is the uniaxial tensile yield stress. From eqn (40) we obtain

$$n_i p_i = n_1^2 + \frac{\nu}{1-2\nu}; \quad p_i q_i = \frac{n_1^2}{1-2\nu} + \frac{\nu^2}{(1-2\nu)^2}. \tag{104}$$

From eqns (4) and (48) it follows that

$$A = H + \frac{2G(1-\nu)}{1-2\nu}; \quad \lambda_3 = 1 + \frac{2G}{A(1-\nu)}\left(n_1^4 - 2n_1^2 - \frac{\nu^2}{1-2\nu}\right). \tag{105}$$

Consequently, use of eqns (51), (52), (97), (104) and (105) results in

$$\frac{\mu_1}{G} = 1 \tag{106}$$

$$\left.\begin{array}{l}\dfrac{\mu_2}{G}\\[2ex] \dfrac{\mu_3}{G}\end{array}\right\} = \frac{1}{2}\left\{\frac{3-4\nu}{1-2\nu} - \frac{4\left(n_1^2 + \dfrac{\nu^2}{1-2\nu}\right)}{2(1-\nu)+(1-2\nu)\dfrac{H}{G}} \pm \sqrt{\left[\frac{1}{1-2\nu} + \frac{4\left(n_1^2 + \dfrac{\nu^2}{1-2\nu}\right)}{2(1-\nu)+(1-2\nu)\dfrac{H}{G}}\right]^2 - \frac{16\left(n_1^2 + \dfrac{\nu}{1-2\nu}\right)^2}{2(1-\nu)+(1-2\nu)\dfrac{H}{G}}}\right\}. \tag{107}$$

These relations refer to a general stress state satisfying the yield criterion. However, the values of μ_2/G and μ_3/G depend only on ν, n_1^2 and H/G. Similar to the previous case, eqn (107) is illustrated in the polar diagram shown in Fig. 4, where the value $\nu = 0.3$ is adopted.

Figure 4(a) shows the elastic behaviour and Fig. 4(b) the hardening response. According to eqns (83) and (105) diffuse wave motion requires that

$$\lambda_3 = \frac{1-2\nu}{2(1-\nu)} = 1 + \frac{2(1-2\nu)}{(1-\nu)\left[2(1-\nu)+(1-2\nu)\dfrac{H}{G}\right]}\left(n_1^4 - 2n_1^2 - \frac{\nu^2}{1-2\nu}\right). \tag{108}$$

This relation is fulfilled for $n_1^2 = 1$ and $H/G = 2(1-\nu)$, i.e. point P in Fig. 4(c) shows, as expected, a state of diffuse wave motion. It is of considerable interest to note that, in contrast to a von Mises criterion, we now have diffuse wave motion in the hardening regime.

Static bifurcation becomes possible when $\lambda_3 = 0$, which due to eqn (105) yields the relation

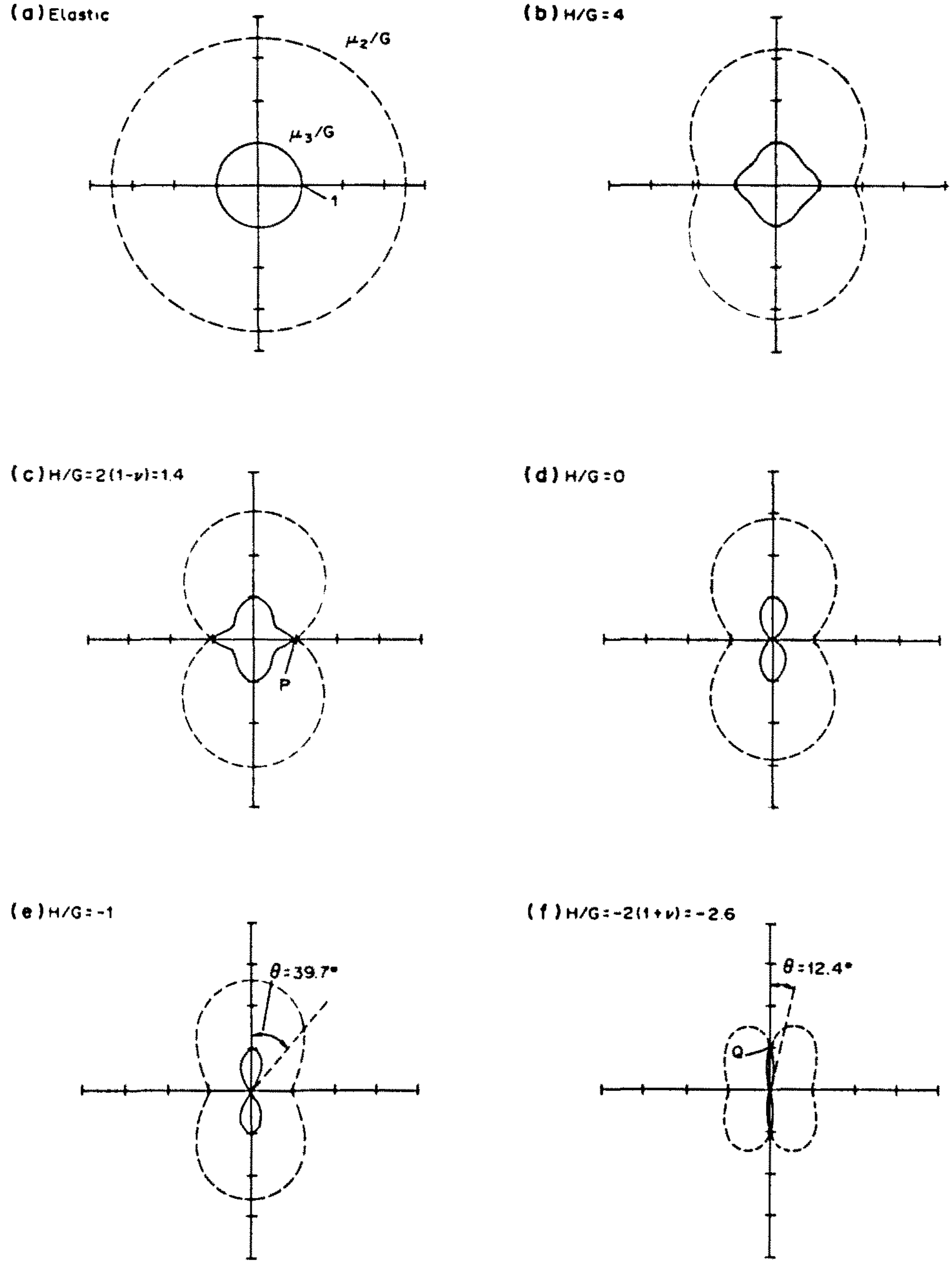

Fig. 4. Rankine plasticity; general stress states; $\nu = 0.3$. Variation of normalized eigenvalues μ_2/G (— — —) and μ_3/G (———) with the value of n_1.

$$n_1^2 = 1 \pm \sqrt{-(1-\nu)\frac{H}{2G}}. \tag{109}$$

The smallest stress level for which this relation becomes possible is when the discriminant is zero, providing the conditions

$$H = 0; \quad n_1^2 = 1 \tag{110}$$

i.e. $\theta = 90^\circ$ as $\sin\theta = n_1$. These expressions are in accordance with the findings by Ottosen and Runesson (1991). It appears that Fig. 4(d) illustrates the situation where static bifurcation first becomes possible.

In the softening region, where $H/G = -1$, the response is shown in Fig. 4(e). Static bifurcation is still possible and it may occur for the value of n_1^2 provided by eqn (109), where it is obvious that only the minus sign in front of the square root can be applied. With $\sin\theta = n_1$, we obtain $\theta = 39.7°$ and in the region defined by $\psi = \pm(90-\theta) = \pm 50.3°$, we have $\lambda_3 < 0$ implying divergence instability as well as negative values of μ_3 not shown in the figure.

Finally, in Fig. 4(f) we obtain a state of diffuse wave motion at point Q. This is in agreement with the fact that, according to Table 1, $n_1^2 = 0$ is a possibility since $f_1 > f_2 = f_3$. As expected, the conditions $n_1^2 = 0$ and $H/G = -2(1+\nu)$ fulfil eqn (108). Like in Fig. 3(f), we observe that the value of the hardening modulus used in Fig. 4(f) can be written as $H = -E$, and this corresponds to a softening branch having a vertical slope; even so, the condition $A = H+2G(1-\nu)/(1-2\nu) = 4G\nu^2/(1-2\nu) > 0$ is fulfilled. The direction for which static bifurcation is possible is determined by $\theta = 12.4°$.

We finally observe from Figs 4(a)–4(c) that $\mu_3/G = 1$ holds for $n_1^2 = 0$ or $n_1^2 = 1$. It appears readily from eqn (107) that this is a general result which holds as long as $H/G \geqslant 2(1-\nu)$. From eqn (107) follows also that for $-2(1+\nu) \leqslant H/G < 2(1-\nu)$ then $n_1^2 = 0$ still implies $\mu_3/G = 1$, whereas $n_1^2 = 1$ now implies $\mu_2/G = 1$, cf. Fig. 4(d)–4(f). This change in behaviour explains the change in the variation of μ_3/G when comparing Figs 4(a)–4(c) with Figs 4(d)–4(f). For the remaining range where $-2(1-\nu)/(1-2\nu) < H/G < -2(1+\nu)$ it is easily shown from eqn (107) that $n_1^2 = 0$ or $n_1^2 = 1$ always implies $\mu_2/G = 1$.

CONCLUSIONS

Explicit analytical expressions were derived for the eigenvalues and eigenvectors of the acceleration wave problem pertinent to general nonassociated plasticity theory. For associated plasticity, these expressions reduce to those obtained by Hill (1962). The eigenvectors are orthogonal for associated plasticity, whereas nonassociated plasticity implies that all eigenvectors are nonorthogonal. It is of interest that one eigenvalue is always equal to the shear modulus. It was shown for a very general class of nonassociated plasticity behaviour that the eigenvalues are always real, implying that so-called "divergence" instability can occur whereas "flutter" instability cannot. In addition, we discovered the interesting phenomenon of diffuse wave motion, which can occur in any elasto-plastic material. In this situation all wave speeds become identical and equal to the elastic distortion wave speed, and the corresponding eigenvectors are arbitrary. Finally, for the models of von Mises and Rankine some of the different phenomena were illustrated graphically, and it was observed that diffuse wave modes might occur both in the hardening and softening regimes.

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APPENDIX

According to eqn (79), we shall determine extrema for the function φ given by

$$\varphi = RS - K_1 n_1^2 - K_2 n_2^2 - K_3 n_3^2 \tag{A1}$$

subjected to the constraint

$$n_1^2 + n_2^2 + n_3^2 - 1 = 0. \tag{A2}$$

In eqn (A1), we have adopted the notation

$$R = f_1 n_1^2 + f_2 n_2^2 + f_3 n_3^2 + \gamma f_c; \quad S = g_1 n_1^2 + g_2 n_2^2 + g_3 n_3^2 + \gamma g_c \tag{A3}$$

$$K_i = (f_i + \gamma f_c)(g_i + \gamma g_c) \quad \text{(no summation)}. \tag{A4}$$

Using the Lagrange multiplier technique, we define

$$L = \varphi - \lambda(n_1^2 + n_2^2 + n_3^2 - 1) \tag{A5}$$

for which the necessary conditions of extremum points are

$$\frac{\partial L}{\partial n_i} = 2A_i n_i = 0 \,\text{(no summation)}; \quad \frac{\partial L}{\partial \lambda} = -(n_1^2 + n_2^2 + n_3^2 - 1) = 0 \tag{A6}$$

where

$$A_i = f_i S + g_i R - K_i - \lambda \quad (i = 1, 2, 3). \tag{A7}$$

In the following we shall assume that $f_1 \geq f_2 \geq f_3$ and $g_1 \geq g_2 \geq g_3$. We may now identify three cases for which eqn (A6) is fulfilled: (i) none of n_1, n_2, n_3 is zero, (ii) one of n_1, n_2, n_3 is zero and (iii) two of n_1, n_2, n_3 are zero.

(i) *None of* n_1, n_2, n_3 *is zero*

From eqn (A6), we conclude that $A_1 = A_2 = A_3 = 0$. From eqn (A7), we determine λ from $A_1 = 0$ and use this expression in conjunction with $A_2 = 0$ and $A_3 = 0$ to obtain

$$\begin{aligned} S(f_1 - f_2) + R(g_1 - g_2) + K_2 - K_1 &= 0 \\ S(f_1 - f_3) + R(g_1 - g_3) + K_3 - K_1 &= 0. \end{aligned} \tag{A8}$$

Using the constraint condition $n_1^2 = 1 - n_2^2 - n_3^2$ in the expressions for R and S given by eqn (A3), we obtain from (A8)

$$\begin{bmatrix} 2(f_1 - f_2)(g_1 - g_2) & (f_1 - f_3)(g_1 - g_2) + (f_1 - f_2)(g_1 - g_3) \\ (f_1 - f_3)(g_1 - g_2) + (f_1 - f_2)(g_1 - g_3) & 2(f_1 - f_3)(g_1 - g_3) \end{bmatrix} \begin{bmatrix} n_2^2 \\ n_3^2 \end{bmatrix} = \begin{bmatrix} (f_1 - f_2)(g_1 - g_2) \\ (f_1 - f_3)(g_1 - g_3) \end{bmatrix}. \tag{A9}$$

Let us now evaluate solutions of eqn (A9) recalling that $n_1^2 > 0$, $n_2^2 > 0$ and $n_3^2 > 0$ hold.

(i1) *Assume* $f_1 > f_2 \geq f_3$, $g_1 > g_2 \geq g_3$. By a row operation and some algebra, eqn (A9) can be transformed to the following form

$$\begin{bmatrix} 2(f_1 - f_2)(g_1 - g_2) & (f_1 - f_3)(g_1 - g_2) + (f_1 - f_2)(g_1 - g_3) \\ 0 & P \end{bmatrix} \begin{bmatrix} n_2^2 \\ n_3^2 \end{bmatrix} = \begin{bmatrix} (f_1 - f_2)(g_1 - g_2) \\ Q \end{bmatrix} \tag{A10}$$

where

$$P = -[(f_1 - f_2)(g_1 - g_3) - (f_1 - f_3)(g_1 - g_2)]^2 \leq 0 \tag{A11}$$

$$Q = (f_1 - f_2)(g_1 - g_2)[(f_2 - f_3)(g_1 - g_3) + (f_1 - f_3)(g_2 - g_3)]^2 \geq 0. \tag{A12}$$

Clearly, in order that $Pn_3^2 = Q$ admits a solution $n_3^2 > 0$, we must require that $P = Q = 0$. As $f_1 > f_2$ and $g_1 > g_2$ the requirement $Q = 0$ implies that the contribution from the bracket present in the expression for Q must be zero. However, as both terms in this bracket are non-negative, we must require $f_2 = f_3$ and $g_2 = g_3$. This requirement also implies that $P = 0$. Therefore we need only consider the case:

(i1a) $f_1 > f_2 = f_3, g_1 > g_2 = g_3$

From the first row in eqn (A10) we obtain

$$n_2^2 + n_3^2 = \tfrac{1}{2}, \quad n_1^2 = \tfrac{1}{2} \tag{A13}$$

and from eqn (A1), the corresponding extremum value of φ becomes

$$\varphi = -\tfrac{1}{4}(f_1 - f_3)(g_1 - g_3) < 0. \tag{A14}$$

(i2) *Assume* $f_1 = f_2 \geq f_3, g_1 > g_2 \geq g_3$. The equation system (A9) reduces to

$$\begin{bmatrix} 0 & (f_1 - f_3)(g_1 - g_2) \\ (f_1 - f_3)(g_1 - g_2) & 2(f_1 - f_3)(g_1 - g_3) \end{bmatrix} \begin{bmatrix} n_2^2 \\ n_3^2 \end{bmatrix} = \begin{bmatrix} 0 \\ (f_1 - f_3)(g_1 - g_3) \end{bmatrix}. \tag{A15}$$

In order that $n_2^2 > 0$ we must require $f_1 = f_2 = f_3$ and, therefore, we only consider the case:

(i2a) $f_1 = f_2 = f_3, g_1 > g_2 \geq g_3$

The equations above are fulfilled identically and we have the solution

$$n_1^2 + n_2^2 + n_3^2 = 1. \tag{A16}$$

In this case, it is easily shown that eqn (A1) yields

$$\varphi = 0. \tag{A17}$$

(i3) $f_1 > f_2 \geq f_3, g_1 = g_2 \geq g_3$. Equation (A9) reduces to

$$\begin{bmatrix} 0 & (f_1 - f_2)(g_1 - g_3) \\ (f_1 - f_2)(g_1 - g_3) & 2(f_1 - f_3)(g_1 - g_3) \end{bmatrix} \begin{bmatrix} n_2^2 \\ n_3^2 \end{bmatrix} = \begin{bmatrix} 0 \\ (f_1 - f_3)(g_1 - g_3) \end{bmatrix}. \tag{A18}$$

Clearly, in order that $n_2^2 > 0$ we must require $g_1 = g_2 = g_3$, i.e. we are left with the case:

(i3a) $f_1 > f_2 \geq f_3, g_1 = g_2 = g_3$

Equation (A18) is always fulfilled and we have

$$n_1^2 + n_2^2 + n_3^2 = 1 \tag{A19}$$

as well as

$$\varphi = 0. \tag{A20}$$

(i4) $f_1 = f_2 \geq f_3, g_1 = g_2 \geq g_3$. Equation (A9) becomes

$$2(f_1 - f_3)(g_1 - g_3)n_3^2 = (f_1 - f_3)(g_1 - g_3). \tag{A21}$$

(i4a) $f_1 = f_2 > f_3, g_1 = g_2 > g_3$

Equation (A21) provides

$$n_3^2 = \tfrac{1}{2}; \quad n_1^2 + n_2^2 = \tfrac{1}{2} \tag{A22}$$

Evaluation of eqn (A1) gives

$$\varphi = -\tfrac{1}{4}(f_1 - f_3)(g_1 - g_3) < 0. \tag{A23}$$

(i4b) $f_1 = f_2 = f_3, g_1 = g_2 \geq g_3$

Equation (A21) is fulfilled identically and we just have

$$n_1^2 + n_2^2 + n_3^2 = 1 \tag{A24}$$

as well as

$$\varphi = 0. \tag{A25}$$

(i4c) $f_1 = f_2 > f_3, g_1 = g_2 = g_3$

The solutions given by eqns (A24) and (A25) are valid.

(ii) *One of* n_1, n_2, *is zero*

(ii1) $n_1 = 0$ $(n_2 \neq 0, n_3 \neq 0)$. From eqn (A6), we obtain $A_2 = A_3 = 0$, which due to eqn (A7) results in

$$f_2 S + g_2 R - K_2 - \lambda = 0$$
$$f_3 S + g_3 R - K_3 - \lambda = 0.$$

Eliminating λ and using the constraint condition $n_2^2 = 1 - n_3^2$, we obtain the relation

$$2(f_2-f_3)(g_2-g_3)n_3^2 = (f_2-f_3)(g_2-g_3). \tag{A26}$$

(ii1a) *Assume* $f_1 \geq f_2 > f_3$, $g_1 \geq g_2 > g_3$

The equation above yields

$$n_3^2 = \tfrac{1}{2}, \quad n_2^2 = \tfrac{1}{2}, \quad n_1^2 = 0 \tag{A27}$$

and φ given by eqn (A1) becomes

$$\varphi = -\tfrac{1}{4}(f_2-f_3)(g_2-g_3) < 0. \tag{A28}$$

(ii1b) *Assume* $f_1 \geq f_2 = f_3$ *and/or* $g_1 \geq g_2 = g_3$

Equation (A26) is identically satisfied and we have simply

$$n_2^2+n_3^2 = \tfrac{1}{2}, \quad n_1^2 = 0 \tag{A29}$$

as well as

$$\varphi = 0. \tag{A30}$$

(ii2) $n_2 = 0$ ($n_1 \neq 0, n_3 \neq 0$). From eqn (A6), we obtain $A_1 = A_3 = 0$ which due to eqn (A7) results in

$$f_1 S+g_1 R-K_1-\lambda = 0$$
$$f_3 S+g_3 R-K_3-\lambda = 0.$$

Eliminating λ and using the constraint condition $n_1^2 = 1-n_3^2$, we obtain

$$2(f_1-f_3)(g_1-g_3)n_3^2 = (f_1-f_3)(g_1-g_3). \tag{A31}$$

(ii2a) *Assume* $f_1 > f_3$, $g_1 > g_3$

The above equation results in

$$n_3^2 = \tfrac{1}{2}, \quad n_2^2 = 0, \quad n_1^2 = \tfrac{1}{2} \tag{A32}$$

and φ given by eqn (A1) becomes

$$\varphi = -\tfrac{1}{4}(f_1-f_3)(g_1-g_3) < 0. \tag{A33}$$

(ii2b) *Assume* $f_1 = f_2 = f_3$, *and/or* $g_1 = g_2 = g_3$

Equation (A31) is identically satisfied and we are left with

$$n_1^2+n_3^2 = 1, \quad n_2^2 = 0, \quad \varphi = 0. \tag{A34}$$

(ii3) $n_3 = 0$ ($n_1 \neq 0, n_2 \neq 0$). From eqn (A6) we obtain $A_1 = A_2 = 0$ which due to eqn (A7) results in

$$f_1 S+g_1 R-K_1-\lambda = 0$$
$$f_2 S+g_2 R-K_2-\lambda = 0.$$

Eliminating λ and using the constraint condition $n_2^2 = 1-n_1^2$, we obtain

$$2(f_1-f_2)(g_1-g_2)n_1^2 = (f_1-f_2)(g_1-g_2). \tag{A35}$$

(ii3a) *Assume* $f_1 > f_2 \geq f_3$, $g_1 > g_2 \geq g_3$

The above equation results in

$$n_1^2 = \tfrac{1}{2}, \quad n_2^2 = \tfrac{1}{2}, \quad n_3^2 = 0 \tag{A36}$$

and φ given by eqn (A1) becomes

$$\varphi = -\tfrac{1}{4}(f_1-f_2)(g_1-g_2) < 0. \tag{A37}$$

(ii3b) *Assume* $f_1 = f_2 \geq f_3$ *and/or* $g_1 = g_2 \geq g_3$

Equation (A35) is, again, identically satisfied and we are left with

$$n_1^2+n_2^2 = 1, \quad n_3^2 = 0, \quad \varphi = 0. \tag{A38}$$

(iii) *Two of* n_1, n_2, n_3 *are zero*

Assume that $n_1^2 = 1$ while $n_2^2 = n_3^2 = 0$. It then immediately follows from eqn (A1) that $\varphi = 0$. Likewise, when $n_2^2 = 1$ or $n_3^2 = 1$ we obtain that $\varphi = 0$.

In conclusion, it has been proved that $\varphi \leq 0$ for all extrema, i.e. $\varphi \leq 0$ holds always. The situations for which $\varphi = 0$ are summarized in Table 1.